

Explicit Solutions to the Continuous Time Albert-Barabási Scale-Free Model.[☆]

Benjamin R. Morin^{a,1}

^a*School of Human Evolution and Social Change, Arizona State University, Tempe, AZ 85287*

Abstract

Presented here is the explicit solution to the continuous time approximation of the Albert-Barabási scale-free network model at any time t . The solution is found by recursively solving the differential equations via integrating factors, identifying a pattern for the coefficients and then proving this observed pattern using induction.

Keywords: Scale-free network, Differential equations, Inductive solution

1. Introduction

The Albert-Barabási scale-free network model has received a great deal of attention for well over a decade since the preliminary work, focused on determining the mechanism that producing power-law degree distribution of certain real-world networks, was written [1]. Work rapidly ensued in the application of such network structure to phenomena such as personal prefer-

[☆]The author would like to thank Sidney Redner for his suggestions that broadened the discussion and intellectual merit of the paper.

Email address: `brmorin@asu.edu` (Benjamin R. Morin)

¹The author is also a member of the Mathematical and Computational Modeling Sciences Center, Arizona State University, PO Box 871904, Tempe, AZ, 85287.

ences, language structure, statistical physics (Bose-Einstein condensation), epidemics and a veritable wealth of biological examples [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The intrinsic concept of preferential attachment, previously called cumulative advantage, was first proposed by Price as an application to citation networks with a slightly more general model [12]. Despite the decades that this concept has persisted in the literature, an explicit solution to the *degree distribution* of such a network has been absent.

One such method used to describe the dynamic evolution of such a network is the doubly indexed, by degree and time, difference equation [13]

$$p_{k,t}(t_i) = \frac{k-1}{2(t-1)} p_{k-1,t-1}(t_i) + \left(1 - \frac{k}{2(t-1)}\right) p_{k,t-1}(t_i).$$

In this model new nodes are introduced at unit times with a single edge to connect preferentially to highly connected existing nodes. The solution to this model was found asymptotically, i.e. as $t \rightarrow \infty$, by Dorogovtsev *et al.* [14]. The continuous time analog was simultaneously solved by Krapivsky *et al.*, again in the “most interesting asymptotic regime ($t \rightarrow \infty$)” [15]. It is unclear if Krapivsky and his co-authors found the explicit time t solution for this paper specifically. Similar later work by Krapivsky and Redner does indeed calculate the explicit solution for the degree distribution for a model where rewiring of connections and a more general preferential structure is modeled [16]. In this, and a previous paper ([17]), the author’s investigate various correlations, e.g. between age of nodes and their connectivity, in addition to utilizing rate equations (differential equations) and the generating function (partial differential equations) to compute the form and various moments of the degree distribution.

Exact solutions for the robust metrics (e.g. degree distribution, mean path length, eigenvalue structure, etc...) are important in the analysis of the structure of the networks themselves or phenomena spreading on them. Indeed, Dorogovtsev *et al.* explicitly calculated the path length of a small world network structure, [18], from the discrete model, and Newman *et al.* did the same to the mean field model [19]. Wang *et al.* address epidemic spread on complex networks without addressing the specific network archetype considered [20]. Instead, they analyze the epidemic threshold produced via the eigenvalue structure of complex topology. Valente has proposed a technique for investigating thresholds of spreads on social networks (either established or as they form) based on the type of adoption employed in the diffusion of innovation by individuals [21]. While he does not explicitly address the network topology itself, it is clear that a need for the distribution on the number of connections, as well as their “quality”, at any time during the spread is important to understanding the resultant diffusion.

Presented here is something of a side-generalization of the scale-free model that was previously solved asymptotically by Krapivsky *et al.*, again explicitly by Krapivsky and Redner and in other approximate forms by Barabási, Albert and Jeong ([22]):

$$\dot{N}_k(t) = \Lambda\delta_{k,m} + \Lambda m [\pi_{k-1}(t)N_{k-1}(t)(1 - \delta_{k,m}) - \pi_k(t)N_k(t)]. \quad (1)$$

I allow new nodes to enter the network at constant rate Λ , and connect to m existing nodes with density dependent preference $\pi_k(t) = \frac{k}{D(t)}$, where $D(t) = 2\Lambda mt + D_0$ is twice the count of edges within the network with $D_0/2$

initial edges. This is the so-called rate equation(s) for the counts of nodes with degree k . The derivation of such an equation is well handled in a number of review papers [23, 24, 25]. When considering the previously solved model equivalence is obtained when there is no rewiring ($r = 0$), the kernel is strictly the Albert-Barabási preferential attachment ($\lambda = 0$), both the node introduction rate and minimum degree are unit ($\Lambda = m = 1$) and when the initial condition is described by two nodes connected with a single edge ($N_0 = D_0 = 2$ and $N_\ell(0) = 2\delta_{\ell,m}$). From this description one can see that the model considered here is dynamically less-rich (the process of rewiring and of more general attachment is excluded) but considers a more general “introduction regime” (the newly introduced nodes are a bit more general).

A common problem with the application of scale-free networks is that while it does match the form of many established structures, i.e. the World Wide Web ([26]), it does not properly match the evolution of such structures [27]. The goal here is to produce explicit, time t , solutions which may be used in addressing questions which take place during the evolution of such network, e.g. synchronicity or “worst-case” spread of disease on networks as considered in [28] and [4] respectively. The ability to compare these time t solutions with specific application data may allow for researchers to more accurately propose better models for the phenomena in question.

In Section 2 I will go over the general recursive form of the solution found via integrating factors. A few base cases, for $k = m$, $m + 1$, and $m + 2$ are worked out in Section 3, and a proof by induction of the recognized form

is given in Section 4. The paper ends with a discussion of the decay of dependence on initial conditions and a few “ tweaks ” to and critiques of the model are suggested that may lead to better approximations.

2. General Solution Form

For the sake of space we define the “ double-count ” of all edges in the network at time t by $D(t) := 2\Lambda m t + D_0$, the total number of nodes as $N(t) := \Lambda t + N_0$, and let $R_k(t) := \pi_k(t)\Lambda m$. Further note that

$$e^{\int R_k(t)dt} = D(t)^{\frac{k}{2}} =: H_k(t).$$

The general solution to $N_k(t)$ can thus be given recursively as

$$\dot{N}_k(t) + R_k(t) = \Lambda \delta_{k,m} + R_{k-1}(t)N_{k-1}(t)(1 - \delta_{k,m}), \quad (2)$$

$$[N_k(t)\dot{H}_k(t)] = (\Lambda \delta_{k,m} + R_{k-1}(t)N_{k-1}(t)(1 - \delta_{k,m})) H_k(t), \quad (3)$$

and finally

$$N_k(t) = \frac{\int [\Lambda H_k(t)\delta_{k,m} + R_{k-1}(t)N_{k-1}(t)H_k(t)(1 - \delta_{k,m})] dt + C_k}{H_k(t)}, \quad (4)$$

$$= \frac{H_2(t)}{m(k+2)}\delta_{k,m} + \frac{\int R_{k-1}(t)N_{k-1}(t)H_k(t)(1 - \delta_{k,m})dt + C_k}{H_k(t)}, \quad (5)$$

where the final step follows from the two observations (when $k \neq -2$)

$$\begin{aligned} \int H_k(t)dt &= \int (2\Lambda m t + D_0)^{\frac{k}{2}} dt, \\ &= \frac{(2\Lambda m t + D_0)^{\frac{k+2}{2}}}{\Lambda m(k+2)} = \frac{H_{k+2}(t)}{\Lambda m(k+2)}, \end{aligned} \quad (6)$$

and

$$\frac{H_{k+n}(t)}{H_k(t)} = \frac{(2\Lambda m t + D_0)^{\frac{k+n}{2}}}{(2\Lambda m t + D_0)^{\frac{k}{2}}} = H_n(t). \quad (7)$$

The constants of integration, C_k , are defined by

$$C_k = N_k(0) - \frac{H_2(0)}{m(k+2)}\delta_{k,m} - \left. \frac{\int R_{k-1}(t)N_{k-1}(t)H_k(t)(1-\delta_{k,m})dt + C_k}{H_k(t)} \right|_{t=0}.$$

These expressions have the general forms (proven later by induction)

$$N_k(t) = \frac{(m+1)\Gamma(k)H_2(t)}{\Gamma(k+3)} + \sum_{i=m}^k \binom{k-1}{i-1} \frac{C_i}{H_i(t)}, \quad (8)$$

with the constants of integration

$$C_i = H_i(0) \left(\frac{(-1)^{i-m+1}\Gamma(i)H_2(0)}{\Gamma(i-m+1)\Gamma(m+1)(i+2)} + \sum_{\ell=m}^i \binom{i-1}{\ell-1} (-1)^{i-\ell} N_\ell(0) \right). \quad (9)$$

The typically assumed initial conditions are $N(0) = m+1$ and $D(0) = 2m$ (i.e. $N_k(0) = \delta_{k,m}$). These expressions for $N_k(t)$ and C_i condense to a closed form solution in terms of a hypergeometric function, ${}_2F_1$, (presented without derivation). Defining $G(t) := \frac{H_1(t)}{H_1(0)} = \sqrt{\frac{2\Delta mt}{D_0} + 1}$ the solution to $N_k(t)$ is

$$N_k(t) = \frac{(m+1)\Gamma(k)H_2(t)}{\Gamma(k+3)} - \frac{A_k(t)}{G^m(t)} + \frac{(G(t)-1)^{k-m}\Gamma(k)}{G^k(t)\Gamma(k-m+1)}, \quad (10)$$

where

$$A_k(t) = \frac{\Gamma(k) {}_2F_1 \left(m+2, m-k; m+3; \frac{1}{G(t)} \right)}{(m+2)\Gamma(m+1)\Gamma(k-m+1)}. \quad (11)$$

It is relatively straightforward to verify this result against that of Krapivsky and Redner ([16]) using Equations 8 and 9 with $m = 1$, $H_k(t) = (2t+2)^{\frac{k}{2}}$, and $N_\ell(0) = 2\delta_{\ell,m}$ to arrive at

$$N_k(t) = \frac{4(t+1)}{k(k+1)(k+2)} + \frac{1}{(t+1)^{\frac{1}{2}}} \sum_{j=0}^{k-1} \frac{\Gamma(k)}{\Gamma(k-j)} \frac{(-1)^j(2j+4)}{j!(j+3)(t+1)^{\frac{j}{2}}}.$$

3. Base Cases

In order to prove the previous solution expressions I employ a recursive/inductive method. The solution for the smallest degree, m , is calculated rather easily on its own with an integrating factor. The degrees $k > m$ require the solution to the previous degree (recursion). A pattern is found through inspection and then showed to hold in general in the inductive step in Section 4. A “trick” is highlighted in the case of $N_{m+2}(t)$ which is employed to preserve the pattern.

3.1. $N_{m+j}(t)$, $j = 0, 1, 2$

In this Subsection I consider only the form of the state variable solution. The form constants of integration are treated in the following Subsection.

3.1.1. $j = 0$

The expression for $\dot{N}_m(t)$ is given by

$$\dot{N}_m(t) = \Lambda - R_m(t)N_m(t) = \Lambda - \Lambda m \pi_m(t)N_m(t).$$

The only additive term is the constant node introduction rate Λ because each new node enters with degree m and thus there may not exist a node of degree less than m to “feed into” $N_m(t)$. The solution follows straightforwardly from Equation 5:

$$N_m(t) = \frac{\Lambda \int H_m(t)dt + C_m}{H_m(t)}, \quad (12)$$

$$= \frac{\Lambda \frac{H_{m+2}(t)}{\Lambda m(m+2)} + C_m}{H_m(t)}, \quad (13)$$

$$= \frac{1}{m(m+2)} H_2(t) + \frac{C_m}{H_m(t)}. \quad (14)$$

3.1.2. $j = 1$

The solution to $N_{m+1}(t)$ depends on $N_m(t)$, since a new node connecting to a node of degree m creates a node of degree $m + 1$. Thus, again from Equation 5, we have a solution in the form:

$$N_{m+1}(t) = \frac{\Lambda m \int \pi_m(t) N_m(t) H_{m+1}(t) dt + C_{m+1}}{H_{m+1}(t)}, \quad (15)$$

$$= \frac{\Lambda m \int \frac{m}{H_2(t)} \left(\frac{H_2(t)}{m(m+2)} + \frac{C_m}{H_m(t)} \right) H_{m+1}(t) dt + C_{m+1}}{H_{m+1}(t)}. \quad (16)$$

Considering just the integral we may employ the properties Equation 6 and 7 to write it as

$$\int \left(\frac{H_{m+1}(t)}{m+2} + \frac{mC_m}{H_1(t)} \right) dt = \frac{H_{m+3}(t)}{\Lambda m(m+2)(m+3)} + \frac{C_m}{\Lambda} H_1(t). \quad (17)$$

Plugging Equation 17 into 16 results in

$$N_{m+1}(t) = \frac{1}{(m+2)(m+3)} H_2(t) + \frac{mC_m}{H_m(t)} + \frac{C_{m+1}}{H_{m+1}(t)}. \quad (18)$$

3.1.3. $j = 2$

The solution to $N_{m+2}(t)$ follows in essentially identical fashion to that of $N_{m+1}(t)$ save for one “tricky” step. Within the integral term, as found in Equation 20, the term $m(m+1)C_m$ is written as $m(m+1)C_m H_{1-1}(t)$. This will not alter the resultant value, but it does keep the form of C_{m+i} consistent. This way of *multiplying by 1* has to be employed for each consecutive ODE. The solution via Equation 5 is

$$N_{m+2}(t) = \frac{\Lambda m \int \pi_{m+1}(t) N_{m+1}(t) H_{m+2}(t) dt + C_{m+2}}{H_{m+2}(t)}, \quad (19)$$

$$= \frac{\Lambda m \int \left(\frac{(m+1)H_{m+2}(t)}{(m+2)(m+3)} + m(m+1)C_m H_{1-1}(t) + \frac{(m+1)C_{m+1}}{H_1(t)} \right) dt + C_{m+2}}{H_{m+2}(t)} \quad (20)$$

This produces the solution

$$N_{m+2}(t) = \frac{(m+1)}{(m+4)(m+3)(m+2)} H_2(t) + \frac{(m+1)mC_m}{2H_m(t)} + \frac{(m+1)C_{m+1}}{H_{m+1}(t)} + \frac{C_{m+2}}{H_{m+2}(t)}. \quad (21)$$

3.1.4. The Pattern

Inferring the pattern at this point is perhaps a touch premature. Finding the solution to $N_{m+3}(t)$ makes the formulae evident, but for the sake of space I leave this to the reader and outline the pattern here using results from the $N_{m+3}(t)$ solution. Each solution is composed of essentially two parts: a part that involves the term $H_2(t)$ and another that involves the constants of integration.

$H_2(t)$: The coefficients for the $H_2(t)$ -term follow for $j = 0, 1, 2$, and 3 : $\frac{1}{m(m+2)}$, $\frac{1}{(m+2)(m+3)}$, $\frac{(m+1)}{(m+4)(m+3)(m+2)}$, and $\frac{(m+1)}{(m+5)(m+4)(m+3)}$. The progression is clear when the first two terms are multiplied and divided by $m+1$ to give the general coefficient of the $H_2(t)$ term for $N_k(t)$ of

$$\frac{m+1}{(k+2)(k+1)(k)}. \quad (22)$$

C_{m+i} : The coefficients for the C_{m+i} -term are each divided by the appropriate $H_{m+i}(t)$ term. Furthermore, as is clearer from the solution of $N_{m+3}(t)$ there is also a term in the form of $\binom{k-1}{m+i-1}$ giving a final form for the coefficient of

$$\binom{k-1}{m+i-1} \frac{1}{H_{m+i}(t)}. \quad (23)$$

Putting the two patterns together I infer a solution identical to Equation 8.

3.2. C_{m+i} , $i = 0, 1, 2$

The constants of integration are found rather straightforwardly as

$$\begin{aligned} C_m &= \left(N_m(0) - \frac{H_2(0)}{m(m+2)} \right) H_m(0), \\ C_{m+1} &= \left(N_{m+1}(0) - mN_m(0) + \frac{H_2(0)}{m+3} \right) H_{m+1}(0), \\ C_{m+2} &= \left(N_{m+2}(0) - (m+1)N_{m+1}(0) + \frac{m(m+1)}{2}N_m(0) - \frac{(m+1)H_2(0)}{2(m+4)} \right) H_{m+2}(0). \end{aligned}$$

The sign before the $H_2(0)$ terms is alternating and with a coefficient of $\binom{m+i-1}{i} \frac{1}{m(m+i+2)}$ for C_{m+i} . The coefficient for the $N_{m+\ell}(0)$ terms is $(-1)^{i-\ell} \binom{m+i-1}{m+\ell-1}$ for $\ell = 0, 1, \dots, i$. This gives an expression identical to Equation 9.

4. Inductive Step

To prove that these are indeed the solutions for all $m+j$, $j = 0, 1, 2, \dots$

I will invoke induction. Suppose that for some $j > 2$ it is true that

$$N_{m+j}(t) = \frac{(m+1)\Gamma(m+j)H_2(t)}{\Gamma(m+j+3)} + \sum_{i=m}^{m+j} \binom{m+j-1}{i-1} \frac{C_i}{H_i(t)}.$$

By Equation 5 we have that

$$N_{m+j+1}(t) = \frac{\Lambda m \int \pi_{m+j}(t) N_{m+j}(t) H_{m+j+1}(t) dt + C_{m+j+1}}{H_{m+j+1}(t)}.$$

Considering just the integral, a few cancellations gives the form

$$\frac{(m+1)\Gamma(m+j+1)}{\Gamma(m+j+3)} \int H_{m+j+1}(t) dt + (m+j) \sum_{i=m}^{m+j} \binom{m+j-1}{i-1} C_i \int H_{m+j-i-1}(t) dt.$$

By using the properties of H in Equations 6 and 7 this resolves to

$$\frac{(m+1)\Gamma(m+j+1)}{\Lambda m \Gamma(m+j+4)} H_{m+j+3}(t) + \sum_{i=m}^{m+j} \binom{m+j}{i-1} C_i \frac{H_{m+j-i+1}(t)}{\Lambda m}. \quad (24)$$

Placing this back into the general solution gives

$$N_{m+j+1}(t) = \frac{(m+1)\Gamma(m+j+1)}{\Gamma(m+j+4)} H_2(t) + \sum_{i=m}^{m+j} \binom{m+j}{i-1} \frac{C_i}{H_i(t)} + \frac{C_{m+j+1}}{H_{m+j+1}(t)}, \quad (25)$$

which is identical in form to Equation 5. Thus, the general formula for $N_k(t)$ is correct. Verifying C_i follows straightforwardly from this proof. The derivation of the single formula in terms of the hypergeometric function as in Equation 5 is left to the reader and easily verified with a computer algebra system.

5. Conclusion/ Discussion

One may view the solution given in Equation 8 as containing two parts: the *core dynamics* and the *initial conditions*. The initial conditions are contained in the series

$$\sum_{i=m}^k \binom{k-1}{i-1} \frac{C_i}{H_i(t)},$$

or perhaps more illustratively in

$$\sum_{i=m}^k \frac{\Gamma(k)}{\Gamma(i)\Gamma(k-i+1)} C_i (2\Lambda m t + D_0)^{-\frac{i}{2}}. \quad (26)$$

It is clear here that for the quantity $N_k(t)$ the dependence on initial conditions generated by all $N_j(t)$ with j less than some i decays on the order of $O(t^{-\frac{i}{2}})$.

We may also consider $\frac{N_k(t)}{N(t)}$, the proportion of nodes that have degree k , and find that this both asymptotically agrees with previous findings of the degree distribution which do not consider transitory behavior and with those

found by Krapivsky and Redner under appropriate restrictions. Also, of mathematical interest, I've inadvertently deduced a seemingly novel identity for hypergeometric functions. Given that $N(t) = \Lambda t + N_0 = \sum_{k=m}^{\infty} N_k(t)$ one may conclude for $N_0 = m + 1$, $D_0 = 2m$, and $N_k(0) = \delta_{k,m}$ that

$$(1 + \Lambda t)^{\frac{m}{2}} \frac{\Gamma(m) - m}{(m+2)\Gamma(m+1)} = \sum_{j=0}^{\infty} \frac{\Gamma(j+m)}{\Gamma(j+1)} {}_2F_1 \left(m+2, -j; m+3; \frac{1}{\sqrt{\Lambda t + 1}} \right).$$

Direct application of this time t solution should be useful for two reasons: formation of and dynamics on complex networks. While the exact rules that particular real-world networks observe in their formation is an intractable key to their structure, one may compare such structures (data) with the solution given here to justify if this mechanism is applicable. For dynamics on non-limiting case networks (e.g. diseases through a smaller population, spread of computer viruses on local networks, ideas spreading through *forming* social cliques, etc...) this solution may also be employed to describe the topology of such structures. Results such as those found by Zhao *et. al.*, on the fragility or scale-free networks to attacks on hubs, should be extended to include networks that are evolving over time [29]. It is possible, that since at time t there is a smaller probability of hubs than in the limiting network (i.e. the tail is not fat enough yet) the network is in fact more resilient to cascading failures than the seemingly more robust limiting network.

There are a few flaws with considering these rate equations as a model for the A-B network growth. In an arbitrarily small time interval from $t = 0$ there has been mass shifted into compartments, $N_k(t)$, with arbitrarily large

degree. Indeed, the process only allows for $\lfloor 2\Lambda mt \rfloor + D_0$ edges at any time t , and thus there should be a maximum degree available at any time. Pastor-Satorras and Vespignani do this to some extent for general scale-free networks with the introduction of exponentially decaying tails induced on the degree distribution [30]. This was done in the interest of investigating the epidemic threshold for finite sized networks in order to find structures with noncritical spread despite the scale-free property. This method however would be inadequate for an investigation into the structure of the networks as they evolve since the networks were taken to be static in their work. Instead the “cut-off” could be modeled implicitly with some sort of modified system that permits the correct number of ODEs to be acting at any time t . A delay type differential equation system containing delays similar to $t - \lfloor \frac{k-N_0}{\Lambda} \rfloor$ and initial conditions like $N_k(t) = 0$ for $t \in [-\lfloor \frac{k-N_0}{\Lambda} \rfloor, 0)$ would perhaps do a better job at modeling the transitory dynamics of this system as they posses this sort of activation switch, i.e. they may only start to accept “mass” once the delay exits the initial data. This would force a finiteness that is otherwise lost in the traditional rate equation techniques.

Further work is also open in the gross generalization of this process. One may consider the degree of a new node to be a *distributed parameter*. In other words the degree a node is introduced with is some random variable M with probability mass function p_M . The concept of rewiring may be reintroduced to the model in addition to a suite of more general attachment kernels. Under these generalizations it is very likely that the network may not retain an asymptotic distribution of $P(k) \propto k^{-\gamma}$, especially if the attachment

kernel differs from proportional. However, the true power of the A-B model should be in its flexibility to allow for generalizations that produces richer behavior and thus, farther reaching applications.

References

- [1] A. Barabási, R. Albert, Emergence of scaling in random networks, *Science* 286 (1999) 509.
- [2] G. Bianconi, A. Barabási, Bose-Einstein condensation in complex networks, *Physical Review Letters* 86 (2001) 5632–5635.
- [3] G. Caldarelli, *Scale-Free Networks: Complex webs in nature and technology*, Oxford University Press, USA, 2007.
- [4] G. Chowell, C. Castillo-Chavez, Worst Case Scenarios and Epidemics, *Bioterrorism: mathematical modeling applications in homeland security* 28 (????).
- [5] N. Christakis, J. Fowler, The spread of obesity in a large social network over 32 years, *The New England Journal of Medicine* 357 (2007) 370.
- [6] K. Lewis, J. Kaufman, M. Gonzalez, A. Wimmer, N. Christakis, Tastes, ties, and time: A new social network dataset using Facebook. com, *Social Networks* 30 (2008) 330–342.
- [7] Z. Liu, Y.-C. Lai, N. Ye, Propagation and immunization of infection on general networks with both homogeneous and heterogeneous components, *Phys. Rev. E* 67 (2003) 031911.

- [8] A. E. Motter, A. P. S. de Moura, Y.-C. Lai, P. Dasgupta, Topology of the conceptual network of language, *Phys. Rev. E* 65 (2002) 065102.
- [9] A. E. Motter, T. Nishikawa, Y.-C. Lai, Range-based attack on links in scale-free networks: Are long-range links responsible for the small-world phenomenon?, *Phys. Rev. E* 66 (2002) 065103.
- [10] R. Pastor-Satorras, A. Vespignani, Epidemic spreading in scale-free networks, *Physical review letters* 86 (2001) 3200–3203.
- [11] M. Small, D. Walker, C. Tse, Scale-free distribution of avian influenza outbreaks, *Physical review letters* 99 (2007) 188702.
- [12] D. Price, NETWORKS OF SCIENTIFIC PAPERS., *Science* (New York, NY) 149 (1965) 510.
- [13] M. Newman, A. Barabasi, D. Watts, *The structure and dynamics of networks*, Princeton Univ Pr, 2006.
- [14] S. Dorogovtsev, J. Mendes, A. Samukhin, Structure of growing networks with preferential linking, *Physical Review Letters* 85 (2000) 4633–4636.
- [15] P. Krapivsky, S. Redner, F. Leyvraz, Connectivity of growing random networks, *Physical Review Letters* 85 (2000) 4629–4632.
- [16] P. Krapivsky, S. Redner, Finiteness and fluctuations in growing networks, *Journal of Physics A: Mathematical and General* 35 (2002) 9517.
- [17] P. Krapivsky, S. Redner, Organization of growing random networks, *Physical Review E* 63 (2001) 066123.

- [18] S. Dorogovtsev, J. Mendes, Exactly solvable small-world network, EPL (Europhysics Letters) 50 (2000) 1.
- [19] M. Newman, C. Moore, D. Watts, Mean-field solution of the small-world network model, Physical Review Letters 84 (2000) 3201–3204.
- [20] Y. Wang, D. Chakrabarti, C. Wang, C. Faloutsos, Epidemic spreading in real networks: An eigenvalue viewpoint, Reliable Distributed Systems, IEEE Symposium on 0 (2003) 25.
- [21] T. Valente, Social network thresholds in the diffusion of innovations, Social Networks 18 (1996) 69–89.
- [22] A. Barabási, R. Albert, H. Jeong, Mean-field theory for scale-free random networks, Physica A: Statistical Mechanics and its Applications 272 (1999) 173–187.
- [23] R. Albert, A. Barabási, Statistical mechanics of complex networks, Reviews of modern physics 74 (2002) 47–97.
- [24] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D. Hwang, Complex networks: Structure and dynamics, Physics Reports 424 (2006) 175–308.
- [25] M. Newman, The structure and function of complex networks, SIAM Review 45 (2003) 167–256.
- [26] A. Barabási, R. Albert, H. Jeong, Scale-free characteristics of random networks: the topology of the world-wide web, Physica A: Statistical Mechanics and its Applications 281 (2000) 69–77.

- [27] L. Adamic, B. Huberman, Power-law distribution of the world wide web, *Science* 287 (2000) 2115.
- [28] T. Nishikawa, A. E. Motter, Y.-C. Lai, F. C. Hoppensteadt, Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize?, *Phys. Rev. Lett.* 91 (2003) 014101.
- [29] L. Zhao, K. Park, Y.-C. Lai, Attack vulnerability of scale-free networks due to cascading breakdown, *Phys. Rev. E* 70 (2004) 035101.
- [30] R. Pastor-Satorras, A. Vespignani, Epidemic dynamics in finite size scale-free networks, *Physical Review E* 65 (2002) 035108.